Bore Reconstruction of Tubular Ducts from its Acoustic Input Impedance Curve

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Abstract – In order to get accurate bore information of tubular objects from acoustical input impedance measurements an optimization approach is proposed and investigated. A suitable waveguide model is reviewed and reconstruction results are presented. Sensitivity of input impedance magnitude to bore variation indicates, that axial accuracy can be improved, without increasing the sampling rate. Advantages of the method are its increased numerical stability, its tolerance to white noise on top of measured data, no requirement to measure phase, and the fact, that measured data need not to be contiguous in the frequency domain. Frequency bands, which are difficult to measure can be omitted.

I. INTRODUCTION

Accurate bore information of tubular objects is essential for the analysis of their acoustical characteristics or their fluid propagation properties. Especially if the duct is folded or bent it is often difficult to get its dimensions by mechanical measurements. If the cross-section is not perfectly circular an additional difficulty arises. In this case acoustically or fluid dynamically equivalent circular cross-sections are often required for subsequent mathematical analysis.

Bore reconstruction starting from acoustical measurements is a desirable approach applicable to many technical fields. Of special interest is such an approach in the field of musical acoustics where actual dimensions of wind instruments are required whenever such an instrument is to be documented, analyzed and improved or repaired. Their tubular length of up to 6m and more, together with their complicated 3-dimensional folding and bending as well as their significant deviations from perfectly circular cross-sections almost rules out attempts using caliper and gauges.

In brass wind instruments, e.g. trumpets, properties like intonation, response, efficiency and even sound timbre are strongly related to the acoustical impedance $Z(j\omega)$, measured at the interface between the player’s lips and the instrument’s mouthpiece. It is a complex function of frequency, defined as the quotient of sound pressure $p(j\omega)$ and sound flow $u(j\omega)$ at this point. Air column resonances corresponding to the playable natural tones of the instrument show up as local maxima of the input impedance magnitude. Their position along the frequency axes, quality factor and absolute value together with the group delay at these frequencies correlate with musical features like intonation, response, efficiency and sound timbre.

The input impedance of arbitrary tubular ducts, terminated by a known impedance at the far end, can be calculated by means of acoustic waveguide modelling. A typical model [1] is reviewed below. It describes transmission matrices $A_i(j\omega)$ representing conical or cylindrical waveguide elements and it takes thermo-viscous losses into account.

Any known duct geometry can be decomposed into $n$ such simple elements yielding $n$ complex frequency dependent waveguide transmission matrices. The figure below was already part of a previous publication of the author, but it contained a typo in the definition of the last matrix element which has now been corrected.

$$A_i(j\omega) = \begin{pmatrix} a_{i,11}(j\omega) & a_{i,12}(j\omega) \\ a_{i,21}(j\omega) & a_{i,22}(j\omega) \end{pmatrix} = \begin{pmatrix} \frac{x_{i+1}}{x_i} C - \frac{D}{(1+x_i)} & \frac{x_{i-1}}{x_i} Z_c D \\ \frac{1}{x_i} (D(\frac{x_{i+1}}{x_i} - \frac{1}{(1+x_i)}) + C T L) & \frac{x_{i-1}}{x_i} (C + \frac{D}{x_i}) \end{pmatrix}$$

(1)

where $C = \cosh(\Gamma L)$, $D = \sinh(\Gamma L)$, $\Gamma = k(1.045 r_v^{-1} + j(1 + 1.045 r_v^{-1})$, $k = \frac{\omega}{c}$, $r_v = \sqrt{\frac{\eta c}{\rho}}$, $Z_c = R_0((1 + 0.369 r_v^{-1} - j0.369 r_v^{-1})$, $R_0 = \frac{\rho c}{S}$ with $\rho$ being the equilibrium gas density, $\omega$ the radian frequency, $\eta$ the shear viscosity coefficient, $c$ the speed of sound, $S_m$ the planar cross-sectional area at the center and $S_l$ the spherical area at the input end of the conical element, $x_i$ the radius of the input spherical sector, $x_{i+1}$ the radius of the output spherical sector and $L$ the distance between the two spheres.

This matrix describes the relationship between sound pressure $p$ and sound flow $u$ in front of and after a conical slice by

$$\begin{pmatrix} p_i(j\omega) \\ u_i(j\omega) \end{pmatrix} = A_i(j\omega) \begin{pmatrix} p_{i+1}(j\omega) \\ u_{i+1}(j\omega) \end{pmatrix}$$

(2)

The product $A(j\omega) = \prod_{i=1}^{L} A_i(j\omega)$ gives an expression for the transmission characteristic of the complete instrument. The ratio between sound pressure and sound flow at the open end of the flaring bell is enforced by the termination or radiation impedance $Z_T$ which is the characteristic impedance of the
open mouth of the instrument. It is here modelled by

$$\frac{p_L(j \omega)}{u_L(j \omega)} = Z_T = Z_c \left( \frac{\omega a_L}{2c} \right)^2 + j \left( \frac{0.61 \omega a_L}{c} \right)$$  \hspace{1cm} (3)

with $a_L = \sqrt{\frac{S}{\pi}}$

If this impedance is transformed by the chain matrix $A$ from the open end back to the mouthpiece, then the acoustical input impedance is obtained.

Because of the one-dimensional assumptions higher oscillation modes are not taken into account. It is therefore only valid for frequencies with wavelengths bigger than duct dimensions.

From the model it is already clear, that there is a unique relationship between geometry and input impedance, but unfortunately it is not so easy to invert the relationship in order to calculate geometry from a given (measured) input impedance.

Solving this inverse problem in the context of musical instruments has been tackled by N. Amir and D. Sharp [2][3] starting from the input reflection function measured in the time domain with a non reflecting sound source connected to the near end. A $\delta$ pulse is transmitted into the instrument and all received reflections arriving at the mouth piece level are recorded. A recursive algorithm known as layer peeling algorithm is then applied to the reflection function to get a table with equidistant bore diameters. The length resolution is directly related to the time resolution (sampling frequency) of the reflection function and the speed of sound.

Basically it is possible to calculate the required reflection function once the complex input impedance spectrum has been measured. A transformation of the measured spectrum into the time domain will yield the instruments pulse response, but the reflecting termination at the near end will add periodic components which are not present, when measuring with a matched, non reflecting sound source. With such data the layer peeling algorithm, as it is known today does not work. So another transformation has to be applied first. It is the bilinear transform

$$IIR(e^{i\Theta}) = \frac{1 - Z_n e^{i\Theta}}{1 + Z_n e^{i\Theta}}$$  \hspace{1cm} (4)

which performs a sort of “unrolling” in the spectral domain, removing additional periodic reflections, caused by the near end termination. The inverse Fourier transform of the spectrum $IIR(e^{i\Theta})$, $\Theta$ being the discretized frequency, is the reflection function, which can serve as input to the layer peeling algorithm.

Practical experiments using this procedure have shown, that this approach is not straightforward. The initial transformation is very sensitive to the value of $Z_0$, the characteristic impedance of the entry cross-section. It also requires accurate impedance measurements in the low frequency range, which is difficult to achieve. One will not gain anything from a high resolution in the frequency domain, because this would only increase the reconstruction length, which is already determined by the physical length of the instrument.

A good resolution of the reconstruction grid requires a high sampling rate of the reflection function, but high bandwidth measurements are difficult, first, because at higher frequencies more energy is radiated and less energy is reflected to the input, second, at higher frequencies plane wave assumptions are not longer valid and higher order modes must be taken into account, especially near the sound source, the microphone and when theoretical calibration spectra are applied. On top of that, the recursive layer peeling algorithm exhibits numerical weaknesses and it accumulates any numerical errors as well as errors in its input data.

II. PROPOSED APPROACH

In order to avoid many of the above difficulties of the analytical approach an optimization procedure has been proposed by the author [4]. A computer program, following a reliable optimization strategy [5], modifies a starting geometry continuously in order to improve the matching between its theoretically obtained impedance spectrum and a measured one of an instrument, which is to be reconstructed. Once the matching is achieved, the model geometry should be a reconstruction of the measured instrument.

Because of the averaging nature of the optimization cost function $CF = \sum_{i=1}^{n} w_i |(Z_{meas}(i\Delta f)) - |Z_{theo}(i\Delta f))|^p$ (with $\Delta f = \text{smp}/n_{FFT}$, $n_{smp} = \text{sampling frequency}$, $n_{FFT} = \text{FFT}$ buffer length, $w_i = \text{weighting factors}$, $p = \text{weighting exponent}$, $Z_{meas}$ $Z_{theo}$ being measured and theoretical impedance values), there is a relative insensitivity to white noise. It is also possible to mask certain impedance values, which are known to be uncertain, by zeroing the corresponding weighting factors. This is especially useful below 30Hz, where most small speakers do not generate much output.

Nevertheless it is essential to avoid any systematic measurement inaccuracies. The basic principle of the measurement system BIAS (Brass Instrument Analysis System), which has been used, is shown in figure 1. The instrument or tubular duct is attached to the measuring head by means of a mouthpiece.
or an adapter, which is tightly pressed to the plane, where the high impedance capillary ends and the sound pressure \( p_{\text{inp}} \) is measured using a small microphone. The sound pressure \( p_{\text{src}} \) at the other end of the capillary is measured simultaneously. The speaker in the closed chamber is driven by the sound card of a PC, which also samples the response measured by the microphones.

Any wide band stimulus signal, which contains sufficient sound energy at all frequencies of interest is adequate. Experiments have been made with white noise, MLS, chirp and sweep stimuli, each having its own advantages and drawbacks. Chirp and sweep signals turned out to be most suitable, because they can easily be amplitude modulated in order to balance the sound level over frequency, even when there are strong resonances in the measuring head present.

A frequency domain generated periodic chirp signal with a repetition period of 65536 samples at 24000Hz (about 2.7sec), an envelope compensating for head resonances and a matching FFT buffer size of 65536 samples was used to make all the measurements for this paper. Because of the periodicity of the stimulus chirp a rectangular window can be applied in the time domain. Each object was measured five times and the resulting data were averaged.

Although the capillary has a high acoustic impedance compared to most objects, which are usually measured, it is not possible to neglect it, if results are to be used for bore reconstruction. Calibration of the whole measurement system with reference objects having a fairly well known impedance is important.

Small volumes, which can be theoretically analyzed as very short closed tubes, have their first resonance above the frequency range of interest. Up to 12kHz they represent a fairly flat magnitude and phase characteristic, which is insensitive to temperature over the whole frequency range. The calibration object, which was used, is shown in figure 4. Its theoretical impedance is shown in figure 2 and 3.

For this purpose the capillary is treated as an acoustical wave guide with complex frequency dependent characteristic. By solving the chain matrix system, the input impedance \( Z_{\text{inp}}(e^{j\Theta}) \) (\( \Theta \) being the discretized frequency) can be derived from the sound pressure quotient \( \frac{p_{\text{inp}}(e^{j\Theta})}{p_{\text{src}}(e^{j\Theta})} \) and two matrix elements \( A_{11}(e^{j\Theta}) \) and \( A_{12}(e^{j\Theta}) \) by

\[
Z_{\text{inp}}(e^{j\Theta}) = \frac{A_{12}(e^{j\Theta})p_{\text{src}}(e^{j\Theta})}{1 - A_{11}(e^{j\Theta})p_{\text{src}}(e^{j\Theta})}
\]

(5)

The two frequency dependent complex vectors \( A_{11}(e^{j\Theta}) \) and \( A_{12}(e^{j\Theta}) \) can be derived from two calibration measurements of known objects. \( R_i \) are the measured complex pressure spectrum ratios \( \frac{p_{\text{inp}}}{p_{\text{src}}} \) and \( Z_i \) the known complex impedance spectra of the two calibration volumes \( V_1 \) and \( V_2 \) (all are \( f(e^{j\Theta}) \)).

\[
A_{11} = \frac{R_1Z_2 - R_2Z_1}{R_1R_2(Z_2 - Z_1)} \quad A_{12} = \frac{Z_1Z_2(R_2 - R_1)}{R_1R_2(Z_2 - Z_1)}
\]

(6)

The one dimensional theory of tubes seems to be satisfactorily accurate for the calibration volumes up to frequencies about 10kHz. There \( \frac{1}{2} \approx 17\text{mm}, \) which is already close to the dimensions of the duct. Even below these frequencies it can easily happen, that mechanical vibrations, small leakages, body sound paths or non linear effects in the capillary are present, which cannot be removed by this type of calibration. So care must be taken in order to design the mechanical components properly. Figure 4 shows the BIAS measuring head without the upper part, which is screwed on top of it, to tightly press the objects to the rubber surface. The microphone is centered and three capillary outlets are arranged symmetrically around this center in order to minimize the influence of stimulating or recording higher order modes.

### III. RECONSTRUCTION RESULTS

Starting from input impedance measurements as described above, bore reconstruction of the measured tubular objects has
been attempted. For this purpose an impedance magnitude matching optimization has been set up. The optimization cost function included about 1000 frequency points between 40Hz and 7.5kHz (later 9kHz) with a frequency increment of 10Hz.

It turned out, that it is difficult to verify reconstruction results in many practical cases, because often there is no other way to determine the duct geometry with the required accuracy and reliability. Especially the lead pipe of a brass instrument, which is a very critical part in terms of intonation and response, often undergoes many manual and iterative treatment steps. If after all the trumpet is perfectly in tune and responding, then even the maker himself would often like to know the exact final dimensions in order to reproduce them another time.

Therefore the approach was tested with tubular ducts consisting of well known – mostly cylindrical or conical – sections which have been industrially produced. At least their length and their inlet resp. outlet cross-sections have been verified mechanically. Anyhow, junctions between sections with different diameters already present some open questions concerning leakage and gaps by unperfect fit.

Figure 5 shows the stepped tube system which was reconstructed first in order to test the achievable accuracy. The entry bore of 15mm fit perfectly to the cylindrical projection of the measuring head, so minimum errors caused by the coupling have been expected.

The impedance matching which has been achieved is shown in figure 6. When these measurements have been made, a smoothing filter was applied to the measured curves in order to eliminate noise. Unfortunately the filter also decreased the quality factor of the resonance peaks, which is the reason, that the theoretical peaks are generally higher than the corresponding measured ones. This was not so much a drawback for the optimization, because relative matching has been selected instead of absolute matching. This means, that the magnitude mismatch at all frequency points was normalized \( \left( \frac{|Z_{\text{meas}}|}{|Z_{\text{theor}}|} - 1 \right) \) before summation. This way mismatch at resonance peaks does not contribute too much compared to the remaining mismatch, because of the relatively few frequency points, which are concerned.

Just for comparison, another reconstruction result is presented, which has been obtained by means of the layer peeling algorithm, applied to a reflection function calculated from a measured trumpet input impedance following equation (4) (without up-sampling as explained below). It is shown in figure 7 together with another reconstruction of the same instrument obtained by David Sharp using the pulse reflectometry system introduced in [2].

It can be seen that David’s reconstruction contains much more details, which is clear, when his sampling rate of 48kHz is compared to the 8kHz bandwidth of the impedance measurements. The differences at the left end originates from the different setup. To connect to the BIAS head an adapter was used, which has been roughly reconstructed together with the instrument. For pulse reflectometry the instrument without mouthpiece was smoothly coupled to the source tube.

A recent reconstruction result is shown in figure 8. The reconstructed object is an experimental lead pipe of a Bb–Trumpet. The bold curve represents the dimensions as indicated by the maker. The average deviation between the curves is 0.2mm. The tolerance of the making process itself can be estimated as around 0.1mm.

The biggest deviation is, where the leadpipe is attached to the BIAS system using an adapter. This adapter is similar to a
mouthpiece, but it provides a much smoother and less reflective transition from the 15mm head projection diameter to the entry bore of the lead pipe, which is below 10mm. Without an adapter it would not be possible to attach any tube with an inner diameter less than 15mm hermetically to the head.

It must be admitted, that the adapter is a weak point of the measurement procedure. It provides strong and early reflections in the most critical reconstruction region, it slips into the measured instrument by a somewhat undefined distance and it possibly creates some kind of gap or notch depending on how well it fits. On top of that, there is a critical dependence on the fitting pressure. If the rubber plate is strongly compressed, the volume of the duct is decreased and resonance frequencies will go up.

Therefore a special procedure has been chosen. After careful calibration of the system using the two calibration volumes, the adapter was mounted and a cylindrical reference tube was attached. The combination of adapter and tube was measured and an acoustical equivalent of the adapter was reconstructed by optimization. This reconstructed adapter was then used for the reconstruction of the lead pipes. The achieved impedance matching is shown in figure 9. The mismatch between the theoretical and the measured curve is almost invisible in the plot. The average of the relative mismatch of all 895 frequency lines is about 60ppm, or 40Ω in absolute terms. These numbers also give an indication for the required measurement accuracy.

IV. AXIAL ACCURACY AND SAMPLING RATE

When the reconstruction results from figure 5 and 7 are compared, then it can be observed that the optimized shape exhibits much better axial resolution than the one resulting from the layer peeling algorithm. This is interesting because both reconstructions have been made from frequency domain spectra with comparable bandwidth. This will now be investigated in more detail.

Layer peeling requires a reflection function, which can be measured in the time domain by pulse reflectometry, but which can also be derived from the input impedance spectrum as the inverse fourier transform of IIR according to equation (4). The resolution of the impedance spectrum $\Delta f = 1/T$ can be increased by lengthening the sampling interval $T$. The inverse FFT of a fine spectrum gives a long reflection function. The layer peeling algorithm does not make use of a long input vector. When the far end reflection of the pulse returns to the near end the algorithm will stop. The duct is then reconstructed all the way long to the far end. Each sample of the pulse response reconstructs an axial length $\Delta L = \frac{c}{\Delta f}$. Resolution and therefore axial accuracy can be gained by increasing the sampling rate or by decreasing the speed of sound $c$. This is very clear as far as the reflection function is concerned.

When reconstructing from impedance, things are a bit different. Accuracy and resolution or not directly coupled. The impedance spectrum consists of discrete frequency lines each of them representing a standing wave with a certain wavelength $\lambda_i$ and a corresponding frequency $i\Delta f$ related to the position $i$ of the spectral line. The standing wave itself originates from sound reflections between the ideally reflecting input plane and another reflecting discontinuity somewhere along the axis of the acoustic wave guide. The position of this discontinuity is therefore directly related to the frequency of the standing wave. If it does not coincide with the spectral grid, then any windowing function will fold its sound energy into the near or even farther neighborhood and some information is lost. A good frequency resolution together with a good windowing function therefore preserves the information of the accurate position of reflecting discontinuities along the duct.

The inverse FFT of a high resolution impedance spectrum transforms high resolution now into length rather than time resolution, which we would like to get. The key to get the time resolution back is not to truncate the pulse response but to make use of the multiple and secondary reflections contained in the tail of the long pulse response. If a reflected pulse returns exactly in between two sampling events, the distance of the reflecting discontinuity cannot be determined accurately. But if we wait another round trip of the pulse, it will now accurately coincide with a sampling moment and the information about its original wave length will be accurately available.

And this is exactly what happens when the bilinear transform (4) is applied to the impedance spectrum before the inverse FFT. All the secondary and multiple reflections, which would arrive at the microphone long after the primary reflection of the instrument’s far end has arrived, are folded forth into the primary reflection period. Unfortunately they often cannot contribute, because most people do not up-sample the spectrum before the transform. In order to make use of the available extra information, the spectrum must be up-sampled (zero padded) before the transformation is applied.

This is like measuring frequencies using an oscilloscope. Let’s say we read about one period per division. If we count 9 periods over ten divisions our reading will be ten times more accurate - at least, if we do not round the result back to an integer...
number of divisions. This is, in a way, what happens, if the impedance spectrum is not up-sampled before the transformation.

The optimum up-sampling rate will depend on the damping of the system, the required reconstruction length and the available spectral resolution. If the damping is not very high it makes sense to make use of the full length of the periodic pulse response. If this is eight times longer than the required reconstruction length, then up-sampling by a factor of eight would exactly match the information content of the measurement.

Because of the nature of time to frequency domain transitions periodic events within the recording interval are transformed into discrete spectral lines and single events will produce distributed spectral effects ($\delta$ pulse $\rightarrow$ flat magnitude). This indicates, that bore discontinuities, which are not evenly spaced in very close distances will leave their footprints in the whole impedance spectrum and can be traced accurately even if parts of it are missing. Yet, true resolution cannot be gained, as it is defined as the capability to separate two closely neighbored features. Such a case will only create high frequency spectral lines, which needs an adequate sampling rate in order to contribute to the reconstruction.

V. SENSITIVITY AND BANDWIDTH DEPENDANCE

Anyhow, reconstruction by optimization does make use of all the information contained in a high resolution impedance spectrum, and it tolerates gaps in the measured spectrum. The sensitivity of magnitude matching with respect to variations of single diameter values has been investigated.

For this purpose a simple about 90cm long artificial duct consisting of two cylindrical and one conical segment has been modelled. Like it is done during magnitude matching optimizations, the sum of all absolute values of impedance magnitude differences over a certain frequency band has been calculated and plotted over the position of the modified diameter value. The overall frequency band of 12kHz was split in a low, a medium and a high band and the investigation has been repeated for three different axial resolutions.

It was expected that the effect of a single parameter modification in a fine axial grid would mainly effect high frequency regions of the spectrum while the same done in a course axial grid would be noticeable even in the low frequency band. This behavior would justify the widespread opinion that wide band measurements are required if fine axial resolution is to be achieved.

In figure 11 the results are shown. Diameter steps in a 1cm axial grid have most effect in the medium band but can be traced in any other band, too. The sensitivity is significantly smaller in the vicinity of bore discontinuities. A 4cm axial grid makes the base band the most sensitive region. According to the spectrum of a triangular pulse higher bands are much lesser effected.

VI. REFERENCES